中国科学技术大学物理学院
2019～2020 学年第 2 学期考试试卷
（B 卷）

（装订线内不要答题）

课程名称：热力学与统计物理（A）　　课程代码：________

开课院系：物理学院　　考试形式：闭卷

姓名：________　学号：________　专业：________

【答题中可能用到的数学关系：

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}\ ; \qquad \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\ ;$$

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}\ ; \qquad \int_0^\infty x^2 \ln\left(1 - e^{-x}\right) dx = -\frac{\pi^4}{45}\text{。】}$$

一、　一气体由 N 个质量为 m 的经典粒子组成，粒子处于体积为 V 的容器中。把该气体当成理想气体，忽略粒子之间的相互作用。

1. 求温度为 T 时系统的单粒子配分函数。
2. 求该理想气体的状态方程。
3. 求系统的内能。
4. 求系统的熵。

$$
\begin{aligned}
z &= \int e^{-\beta\varepsilon(\mathbf{p})}\frac{d^3rd^3p}{h^3} = \frac{4\pi V}{h^3}\int_0^\infty p^2 e^{-p^2/(2mk_BT)}dp \\
&= \frac{4\pi V}{h^3}\big(2mk_BT\big)^{3/2}\int_0^\infty x^2e^{-x}dx = V\Big(\frac{2\pi mk_BT}{h^2}\Big)^{3/2}
\end{aligned}
$$

$$
p = \frac{N}{k_BT}\Big(\frac{\partial \ln z}{\partial V}\Big)_T = \frac{Nk_BT}{V}
$$

$$
U = -N\Big(\frac{\partial \ln z}{\partial T}\Big)_V = \frac{3Nk_BT}{2}
$$

$$
\begin{aligned}
S &= Nk_B\ln z + \frac{U}{T} - k_B\ln N! \\
&= Nk_B\ln\Big[V\Big(\frac{2\pi mk_BT}{h^2}^{3/2}\Big)\Big] + \frac{3Nk_B}{2} - k_B(N\ln N - N) \\
&= Nk_B\ln\Big[\frac{V}{N}\Big(\frac{2\pi mk_BT}{h^2}^{3/2}\Big)\Big] + \frac{5Nk_B}{2}
\end{aligned}
$$

二、 同上题，考虑相互作用后，当气体的浓度趋于零时仍然可以当成理想气体；当浓度有限大时，气体的状态方程可以用 van der Waals 方程近似，

$$\left(p+\frac{aN^2}{V^2}\right)(V-Nb)=Nk_BT,$$

其中 a, b 为大于零的常数。利用上题结果。

1. 求系统的内能。
2. 求体系的熵。
3. 求系统的等容热容。
4. 求系统的等压热容。

保持粒子数不变，体积趋于无穷大时恢复到理想气体

$$dF=-SdT-pdV$$

$$\begin{aligned}
&\left(\frac{\partial U}{\partial V}\right)_T=\left(\frac{\partial[F+TS]}{\partial V}\right)_T=\left(\frac{\partial F}{\partial V}\right)_T+T\left(\frac{\partial S}{\partial V}\right)_T=-p+T\left(\frac{\partial p}{\partial T}\right)_V\\
&=-\frac{Nk_BT}{V-Nb}+\frac{aN^2}{V^2}+\frac{Nk_BT}{V-Nb}=\frac{aN^2}{V^2}\\
&U(V,T)=U(\infty,T)+\int_\infty^V a\frac{N^2}{V^2}dV=\frac{3Nk_BT}{2}-\frac{aN^2}{V}
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{\partial S}{\partial V}\right)_T=\left(\frac{\partial p}{\partial T}\right)_v=\left(\frac{\partial}{\partial T}\right)_V\left(\frac{Nk_BT}{V-Nb}-\frac{aN^2}{V^2}\right)=\frac{Nk_B}{V-Nb}\\
&S(V,T)-S(V_0,T)=\int_{V_0}^V\frac{Nk_B}{V-Nb}dV=Nk_B\ln\frac{V-Nb}{V_0-Nb}\\
&S(V,T)=Nk_B\ln\frac{V-Nb}{N}\left(\frac{2\pi mk_BT}{h^2}\right)^{3/2}+\frac{5Nk_B}{2}+\lim_{V_0\to\infty}Nk_B\ln\frac{V_0}{V_0-Nb}\\
&\qquad\qquad=Nk_B\ln\frac{V-Nb}{N}\left(\frac{2\pi mk_BT}{h^2}\right)^{3/2}+\frac{5Nk_B}{2}
\end{aligned}$$

$$C_v=\left(\frac{\partial U}{\partial T}\right)_V=\frac{3Nk_B}{2}$$

$$
\begin{aligned}
p &= \frac{Nk_BT}{V-Nb} - \frac{aN^2}{V^2} \\
C_p &= T\Big(\frac{\partial S}{\partial T}\Big)_p = T\frac{\partial(S,p)}{\partial(T,p)} = T\frac{\partial(S,p)}{\partial(T,V)}\frac{\partial(T,V)}{\partial(T,p)} \\
&= T\Big[\Big(\frac{\partial S}{\partial T}\Big)_V\Big(\frac{\partial p}{\partial V}\Big)_T - \Big(\frac{\partial S}{\partial V}\Big)_T\Big(\frac{\partial p}{\partial T}\Big)_V\Big]\Big(\frac{\partial V}{\partial p}\Big)_T \\
&= T\Big(\frac{\partial S}{\partial T}\Big)_V - T\Big(\frac{\partial S}{\partial V}\Big)_T\Big(\frac{\partial p}{\partial T}\Big)_V\Big(\frac{\partial V}{\partial p}\Big)_T \\
&= C_v - T\Big(\frac{\partial p}{\partial T}\Big)_V^2 \Big/ \Big(\frac{\partial p}{\partial V}\Big)_T \\
&= \frac{3Nk_B}{2} - \frac{(Nk_B)^2T}{(V-Nb)^2}\Big/\Big[-\frac{Nk_BT}{(V-Nb)^2} + \frac{2aN^2}{V^3}\Big] \\
&= \frac{3Nk_B}{2} + \frac{Nk_B}{1-2aN(V-Nb)^2/(k_BTV^3)} \\
&= \frac{5Nk_B}{2} + \frac{2aN(V-Nb)^2}{k_BTV^3 - 2aN(V-Nb)^2}
\end{aligned}
$$

三、 体积为 V 的空腔里的平衡电磁辐射可以当成处于热平衡的无相互作用的光子气体。光子的色散关系为：$\varepsilon(\mathbf{p}) = c|\mathbf{p}|$，其中 $\mathbf{p}$ 为光子动量，c 为真空光速。每个光子可与处于两种偏振状态。

1. 请写出单位体积里光子的态密度。
2. 请证明温度为 T 时，光子气体的内能 $U(T,V) = u(T)V$，其中 $u(T)$ 正比于 T^4。
3. 请证明光子气体的压强 $p(T,V) = u(T)/3$。

$$\Omega(\varepsilon) = 2 \times \int \delta(\varepsilon - cp)\frac{d^3r d^3p}{h^3} = 2V \times 4\pi \int_0^\infty \delta(\varepsilon - cp)p^2 dp = \frac{8\pi V}{c^3h^3}\varepsilon^2$$

$$\begin{aligned} U(T,V) &= \int \Omega(\varepsilon)n(\varepsilon)\varepsilon\, d\varepsilon = \int_0^\infty \frac{8\pi V}{c^3h^3}\frac{\varepsilon^3}{e^{\beta\varepsilon}-1}d\varepsilon \\ &= \frac{8\pi V(k_BT)^4}{c^3}\int_0^\infty \frac{x^3}{e^x-1}dx = \frac{8\pi^5(k_BT)^4}{15c^3h^3} \times V \end{aligned}$$

$$\begin{aligned} \ln\Xi &= -\int \Omega(\varepsilon)\ \ln[1-e^{-\beta\varepsilon}]\, d\varepsilon = -\frac{8\pi V}{c^3h^3}\int \ln[1-e^{-\beta\varepsilon}]\varepsilon^2\, d\varepsilon \\ &= -\frac{8\pi V}{c^3h^3}\frac{\varepsilon^3}{3}\ln[1-e^{-\beta\varepsilon}]\Big|_0^\infty + \frac{8\pi\beta V}{3c^3h^3}\int_0^\infty \frac{\varepsilon^3}{e^{\beta\varepsilon}-1}\ d\varepsilon \\ &= \beta U(T,V)/3 = \frac{u(T)}{3k_BT}V \\ p &= \frac{1}{\beta}\Big(\frac{\partial \ln\Xi}{\partial V}\Big)_\beta = u(T)/3 \end{aligned}$$

四、 某热机以光子气体为工作物质在温度为 T_1 的热源和 T_2 的冷源之间做理想 Carnot 循环。利用上题结果:

1. 在 $p-V$ 图上画出循环过程。
2. 求每个过程对外做功和吸热的量。
3. 求该热机的工作效率。

1.

$$\begin{aligned} U &= u(T)V = aT^4V \qquad\qquad p = \frac{u}{3} = \frac{aT^4}{3} \\ \ln\Xi &= \frac{u(T)}{3k_BT}V = \frac{aT^4}{3k_BT}V \\ S &= k_B\ln\Xi + \frac{U}{T} = \frac{4u}{3T}V = \frac{4a}{3}T^3V \\ const \equiv S &= \frac{4a}{3}\left(\frac{3p}{a}\right)^{3/4}V \qquad\qquad \text{可逆绝热过程} \\ &= b\, p^{3/4}V \Rightarrow p \propto V^{4/3} \end{aligned}$$

- 等温膨胀过程

$$\begin{aligned} W_{12} &= \int_{V_1}^{V_2} pdV = p_1(V_2-V_1) = \frac{a}{3}T_1^4(V_2-V_1) \\ Q_{12} &= \Delta U_{12} + W_{12} = u(T_1)(V_2-V_1) + p_1(V_2-V_1) = \frac{4a}{3}T_1^4(V_2-V_1) \end{aligned}$$

- 绝热膨胀过程

$$\begin{aligned} T_1^3V_2 &= T_2^3V_3 \\ Q_{23} &= 0 \\ W_{23} &= \Delta U_{23} = U(T_2,V_3) - U(T_1,V_2) = a(T_2^4V_3 - T_1^4V_2) \end{aligned}$$

- 等温压缩

$$\begin{aligned} W_{34} &= \int_{V_3}^{V_4} pdV = p_2(V_4-V_3) = \frac{a}{3}T_2^4(V_4-V_3) \\ Q_{34} &= \Delta U_{34} + W_{34} = \frac{4a}{3}T_2^4(V_4-V_3) \end{aligned}$$

- 绝热压缩

$$\begin{aligned} T_1^3V_1 &= T_2^3V_4 \\ Q_{41} &= 0 \\ W_{41} &= \Delta U_{41} = U(T_1,V_1) - U(T_2,V_4) = a(T_1^4V_1 - T_2^4V_4) \end{aligned}$$

2. 对外总做功

$$
\begin{aligned}
W &= W_{12} + W_{23} + W_{34} + W_{41} = \frac{4a}{3}[T_1^4(V_2 - V_1) - T_2^4(V_3 - V_4)] \\
&= \frac{4a}{3}[T_1^4(V_2 - V_1) - T_2 \times T_2^3(V_3 - V_4)] = \frac{4a}{3}[T_1^4(V_2 - V_1) - T_2 \times T_1^3(V_2 - V_1)] \\
&= \frac{4a}{3}(T_1 - T_2)T_1^3(V_2 - V_1) \\
\eta &= \frac{W}{Q_{12}} = \frac{T_1 - T_2}{T_1} = 1 - \frac{T_2}{T_1}
\end{aligned}
$$

五、 一晶格里有 N 个格点，由于相互作用，每个格点上最多只能有一个电子。当格点没有电子时，能量为 0；有电子时能量为 ε。每个电子可以处于自旋向上或者向下两种状态。格点之间距离比较远，电子在格点之间的跃迁可以忽略不计。

1. 求温度为 T，化学势为 μ 时，电子的巨配分函数。
2. 求系统的平均电子数及其电子数的涨落。
3. 求系统的内能和熵。

$$
\Xi = \left[1 + 2e^{-\beta(\varepsilon-\mu)}\right]^N = Z^N
$$

$$
\begin{aligned}
N_e &= \left(\frac{\partial \ln \Xi}{\partial \beta\mu}\right)_\beta = N\left(\frac{\partial \ln Z}{\partial \beta\mu}\right)_\beta \\
&= N\frac{2e^{-\beta(\varepsilon-\mu)}}{1 + 2e^{-\beta(\varepsilon-\mu)}} = \frac{N}{1 + e^{\beta(\varepsilon-\mu)}/2} \\
\Delta N_e^2 &= \frac{\partial^2 \ln \Xi}{\partial(\beta\mu)^2} = \left(\frac{\partial N_e}{\partial \beta\mu}\right)_\beta \\
&= \frac{Ne^{\beta(\varepsilon-\mu)}/2}{[1 + e^{\beta(\varepsilon-\mu)}/2]^2}
\end{aligned}
$$

$$
\begin{aligned}
U &= -\left(\frac{\partial \ln \Xi}{\partial \beta}\right)_{\beta\mu} = \frac{N \times 2\varepsilon e^{-(\beta\varepsilon-\mu)}}{1 + e^{-(\beta\varepsilon-\mu)}} = \frac{N\varepsilon}{1 + e^{\beta(\varepsilon-\mu)}/2} \\
S &= k_B[\ln \Xi + \beta U - \beta\mu N_e] = Nk_B\left\{\ln[1 + 2e^{-\beta(\varepsilon-\mu)}] + \frac{\beta(\varepsilon-\mu)}{1 + e^{\beta(\varepsilon-\mu)}/2}\right\}
\end{aligned}
$$